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The Higgs-Yukawa Two-scale Effective Potential

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Abstract

The Higgs-Yukawa model generically contains two possibly largely differing scales. To resum the corresponding logarithms in the effective potential of the model a two-scale subtraction scheme is employed. The beta functions in this scheme depend on the renormalization scale-ratio and a large log's resummation has to be performed on them. Two partial renormalization group equations are derived and used to compute the two-scale running parameters. Finally, the LO two-scale effective potential is determined and the applicability of decoupling to the present theory is discussed.

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1 Introduction

In their classic paper on spontaneous symmetry breaking in massless field theories Coleman and Weinberg [1] explained how to use the renormalization group (RG) to sum up large logarithms in the effective potential (EP). Surprisingly, this treatment has only been systematically extended to massive theories relatively recently [2]. Provided one takes the running of the cosmological constant into account it is straightforward to sum up logarithms *where there is only one relevant scale* in the problem. However, when there is more than one scale in the problem it is less clear how to proceed.

In the single-scale case one can simply remove large logarithms from the perturbation series by setting the renormalization scale μ equal to the relevant scale. Hence, at this scale one can trust conventional (loopwise) perturbation theory. To recover the effective potential (or more generally the effective action) at any other scale one uses the RG equation. In the multi-scale case there is *no* choice of μ that will simultaneously remove all the logarithms from the perturbation series, and so one does not have a trustworthy boundary condition to RG-evolve from.

In ref. [3] it was argued that one could extend the standard RG equation to include *several* renormalization points. A multi-scale version of $\overline{\text{MS}}$ was defined where *independent* RG scales κ_i were associated with each coupling λ_i. However, there are a number of obstacles to the application of this scheme. The principal problem is the occurrence of logarithmic terms in the RG functions which render the perturbative RG functions useless (a detailed discussion of this type multi-scale scheme is given in [4]). In a previous paper [5] we introduced an alternative minimal multi-scale scheme. As in the scheme of refs. [3] the beta functions contain logarithmic terms. However, within our scheme it is straightforward to implement *a large logarithms resummation on the RG functions* themselves[3].

To illustrate our scheme we used the $O(N)$-symmetric ϕ^4 theory [5]. At first sight this does not look like a two-scale system since there is only one mass parameter in the Lagrangian. However, when computing the EP there are effectively two scales, namely the Higgs and the Goldstone masses. Note that in the EP calculation the Goldstone mass is *not* zero since there is a non-zero external current in the problem (except at the minimum of the potential). As one approaches the tree level minimum in the broken phase the ratio of the Higgs and Goldstone mass becomes large and a two-scale RG is required.

In this paper we apply the method to a more complicated example, the Yukawa-Higgs model. This is a genuine *two-sided* multi-scale system in which the fermion can be more massive than the boson or vice versa (whereas in the $O(N)$-case the Higgs is always heavier than the Goldstone). Moreover this model allows us to see how our two-scale RG fits in with the expected decoupling of heavy particles [6].

The outline of the paper is as follows. In section 2 we review the standard $\overline{\text{MS}}$ RG approach to LO summations. In section 3 we motivate the idea of two-scale renormalization and introduce our minimal two-scale subtraction scheme. In section 4 we compute the LO two-scale RG functions and use the results in section 5 to compute the

[3]Of course, one can in principle perform a large logarithms expansion of the beta functions in the scheme of ref [3]. However, the differential equations one must solve to execute such an expansion seem quite formidable [4].

LO running parameters. In section 6 we finally compute the two-scale RG improved potential to leading order.

2 The one-loop effective potential in $\overline{\text{MS}}$

Let us consider the $O(N)$-symmetric Yukawa model with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - \frac{\lambda}{24}\phi^4 - \frac{1}{2}m^2\phi^2 - \Lambda + \overline{\psi}_a\,(i\partial\!\!\!/ - g\phi)\,\psi_a, \tag{1}$$

where $\psi_a,\ a = 1, ..., N$ is a N-component Dirac field. The N Dirac fermions interact with the $O(N)$-singlet scalar field ϕ via an $O(N)$-invariant Yukawa coupling. Here Λ is a cosmological constant which enters non-trivially in the RG equation for the effective potential.

A loop-wise perturbation expansion of the effective potential [1] yields in the $\overline{\text{MS}}$-scheme to one loop

$$\begin{aligned} V^{(\text{tree})} &= \frac{\lambda}{24}\varphi^4 + \frac{1}{2}m^2\varphi^2 + \Lambda, \\ V^{(\text{1-loop})} &= \frac{\hbar}{(4\pi)^2}\frac{\mathcal{M}_1{}^2}{4}\left(\log\frac{\mathcal{M}_1}{\mu^2} - \frac{3}{2}\right) - \frac{\hbar}{(4\pi)^2}N\,\mathcal{M}_2{}^2\left(\log\frac{\mathcal{M}_2}{\mu^2} - \frac{3}{2}\right), \end{aligned} \tag{2}$$

where

$$\mathcal{M}_1 = m^2 + \tfrac{1}{2}\lambda\varphi^2, \quad \mathcal{M}_2 = g^2\varphi^2, \tag{3}$$

and μ is the $\overline{\text{MS}}$-renormalization scale. The one-loop contribution to the EP thus contains logarithms of the ratios $\mathcal{M}_i/\mu^2$ to the first power and in general the n-loop contribution will be a polynomial of the nth order in these logarithms.

In view of these logarithms the loop-wise expansion may be trusted only in a region in field- and coupling-space where simultaneously

$$\frac{\hbar\lambda}{(4\pi)^2} \ll 1, \quad \frac{\hbar\lambda}{(4\pi)^2}\log\frac{\mathcal{M}_i}{\mu^2} \ll 1; \qquad \frac{\hbar g^2}{(4\pi)^2} \ll 1, \quad \frac{\hbar g^2}{(4\pi)^2}\log\frac{\mathcal{M}_i}{\mu^2} \ll 1. \tag{4}$$

If $\mathcal{M}_2 \gg \mathcal{M}_1$ or $\mathcal{M}_1 \gg \mathcal{M}_2$ these conditions may not be fulfilled even with a judicious choice of μ. Hence, to obtain a sensible range of validity one has to resum *both* logarithms in the EP. In the one-scale case (or if $\log(\mathcal{M}_1/\mathcal{M}_2)$ is not too large) this would be achieved to LO by solving the one-loop $\overline{\text{MS}}$ RG equation for the effective potential and by employing the corresponding tree-level boundary conditions. Here, we have to deal with two relevant scales. The necessary generalization of the $\overline{\text{MS}}$ scheme and the usual RG approach allowing for as many renormalization scales as there are relevant scales in the theory has been given in [5].

3 The minimal two-scale subtraction scheme $\overline{\text{2MS}}$

To track the two differing log's with two corresponding renormalization scales we use the freedom of performing a finite renormalization and add to one loop a *finite* counterterm to the Lagrangian

$$\Delta\mathcal{L} = \frac{\hbar}{2(4\pi)^2}\mathcal{M}_1^2\log\frac{\mu}{\kappa_1} - \frac{\hbar}{2(4\pi)^2}N\,\mathcal{M}_2{}^2\log\frac{\mu}{\kappa_2}, \tag{5}$$

where we have introduced new renormalization scales κ_1, κ_2 to replace the usual $\overline{\rm MS}$ scale μ. Note that $\Delta\mathcal{L}$ is in fact a polynomial of fourth order in the fields consistent with renormalizability.

In the minimal two-scale subtraction scheme $\overline{\rm 2MS}$ thence introduced the one-loop contribution to the EP becomes

$$V^{(\text{1-loop})} = \frac{\hbar}{(4\pi)^2}\frac{\mathcal{M}_1{}^2}{4}\left(\log\frac{\mathcal{M}_1}{\kappa_1{}^2}-\frac{3}{2}\right) - \frac{\hbar}{(4\pi)^2}N\mathcal{M}_2{}^2\left(\log\frac{\mathcal{M}_2}{\kappa_2{}^2}-\frac{3}{2}\right). \tag{6}$$

As we shall see the *higher* loop beta functions inevitably depend on $\log(\kappa_1/\kappa_2)$ and to obtain trustworthy beta functions some resummation of these logarithms is necessary.

The general features to be respected by $\overline{\rm 2MS}$ are:

i) The effective action Γ, when expressed in terms of the $\overline{\rm 2MS}$ parameters, should be independent of the $\overline{\rm MS}$ scale μ.

ii) When $\kappa_1 = \kappa_2$ $\overline{\rm 2MS}$ should coincide with $\overline{\rm MS}$ at that scale.

iii) When $N = 0$ there are no fermions and so the second set of beta functions for λ, m^2, φ Λ and ψ is zero. In the large-N limit there are no Higgs contributions and so the first set of beta functions is zero in this limit.

iv) When $\kappa_i^2 = \mathcal{M}_i$ the standard loop expansion should render a reliable approximation to the full EP insofar as $\frac{\hbar}{(4\pi)^2}\lambda(\kappa_1,\kappa_2)$ and $\frac{\hbar}{(4\pi)^2}g^2(\kappa_1,\kappa_2)$ are "small".

Starting now from the identity

$$\Gamma_{\overline{\rm MS}}[\lambda_{\overline{\rm MS}}, m^2_{\overline{\rm MS}}, \Lambda_{\overline{\rm MS}}, \varphi_{\overline{\rm MS}}, g_{\overline{\rm MS}}, \psi_{\overline{\rm MS}}; \mu] = \Gamma[\lambda, m^2, \Lambda, \varphi, g, \psi; \kappa_1, \kappa_2] \tag{7}$$

we derive the two $\overline{\rm 2MS}$ RGE's corresponding to variations of the scales κ_i, where the other scale κ_j and the $\overline{\rm MS}$ parameters are held fixed, in much the same way as the $\overline{\rm MS}$ RG is usually derived. Specializing to the effective potential we obtain

$$\mathcal{D}_i V = 0, \quad \mathcal{D}_i = \kappa_i\frac{\partial}{\partial\kappa_i} + {}_i\beta_{g^2}\frac{\partial}{\partial g^2} + {}_i\beta_\lambda\frac{\partial}{\partial\lambda} + {}_i\beta_{m^2}\frac{\partial}{\partial m^2} + {}_i\beta_\Lambda\frac{\partial}{\partial\Lambda} + {}_i\beta_\varphi\frac{\partial}{\partial\varphi}. \tag{8}$$

Of course, ψ has to be renormalized also and there are corresponding beta functions ${}_i\beta_\psi$ which are, however, not relevant to effective potential calculations. The two remaining sets of beta functions are defined as usual

$${}_i\beta_{g^2} = \kappa_i\frac{dg^2}{d\kappa_i}, \quad {}_i\beta_\lambda = \kappa_i\frac{d\lambda}{d\kappa_i}, \quad {}_i\beta_{m^2} = \kappa_i\frac{dm^2}{d\kappa_i}, \quad {}_i\beta_\Lambda = \kappa_i\frac{d\Lambda}{d\kappa_i}, \quad {}_i\beta_\varphi = \kappa_i\frac{d\varphi}{d\kappa_i} \tag{9}$$

for $i = 1, 2$. In general they may be functions not only of g^2, λ, m^2 as are the $\overline{\rm MS}$ RG functions but also of $\log(\kappa_1/\kappa_2)$.

Note that property ii) requires the sum of the $\overline{\rm 2MS}$ RG functions at $\kappa_1 = \kappa_2$ to coincide with the $\overline{\rm MS}$ RG function at that scale

$${}_1\beta_.(\kappa_1 = \kappa_2) + {}_2\beta_.(\kappa_1 = \kappa_2) = \beta_{.,\overline{\rm MS}}, \tag{10}$$

where the set of $\overline{\rm MS}$ beta functions is given to one loop by

$$\begin{aligned}
\beta^{(\text{1-loop})}_{g^2,\overline{\rm MS}} &= \frac{\hbar}{(4\pi)^2}(4N+6)g^4, \quad \beta^{(\text{1-loop})}_{\lambda,\overline{\rm MS}} = \frac{\hbar}{(4\pi)^2}\left(3\lambda^2 + 8N\lambda g^2 - 48Ng^4\right),\\
\beta^{(\text{1-loop})}_{m^2,\overline{\rm MS}} &= \frac{\hbar}{(4\pi)^2}\left(\lambda + 4Ng^2\right)m^2, \quad \beta^{(\text{1-loop})}_{\Lambda,\overline{\rm MS}} = \frac{\hbar}{2(4\pi)^2}m^4,\\
\beta^{(\text{1-loop})}_{\varphi,\overline{\rm MS}} &= -\frac{\hbar}{(4\pi)^2}2Ng^2\varphi, \quad \beta^{(\text{1-loop})}_{\psi,\overline{\rm MS}} = \frac{\hbar}{(4\pi)^2}g^2\psi.
\end{aligned} \tag{11}$$

As we want to vary κ_1 and κ_2 independently we must respect the integrability condition

$$[\kappa_1 d/d\kappa_1, \kappa_2 d/d\kappa_2] = [\mathcal{D}_1, \mathcal{D}_2] = 0, \tag{12}$$

which later allows us to resum logarithms in the $\overline{\text{2MS}}$ beta functions. An essential feature of a mass-independent renormalization scheme such as $\overline{\text{MS}}$ is that the beta functions do not depend on the renormalization scale μ. Unfortunately we cannot generalize this to the multi-scale case and demand that the two sets of beta functions be independent of $\log(\kappa_1/\kappa_2)$. In fact, the independence of the RG functions from the scales κ_i, ie. $[\kappa_i\partial/\partial\kappa_i, \mathcal{D}_j] = 0$, is incompatible with the integrability condition (12). However, it is possible to arrange for *one* of the two sets of RG functions to be independent of κ_1/κ_2 [4], ie. we can take the first set of beta functions to be independent of κ_1/κ_2, or more formally

$$[\kappa_i\partial/\partial\kappa_i, \mathcal{D}_1] = 0. \tag{13}$$

Alternatively we can take the second set of RG functions (tracking the fermionic scale) to be independent of κ_1/κ_2, ie.

$$[\kappa_i\partial/\partial\kappa_i, \mathcal{D}_2] = 0. \tag{14}$$

Consider prescription (13) where one takes the first set of beta functions (tracking the Higgs scale) to be independent of $\kappa_/\kappa_2$. Here the ${}_1\beta_.$ will look like the beta functions for a single-scale theory. Now, if the fermion is much heavier than the Higgs ($\mathcal{M}_2 \ll \mathcal{M}_1$) we expect to observe a decoupling of the heavy particle from the low energy theory. Thus the low energy (Higgs) theory *is* a single-scale theory, and so it is natural to identify condition (13) with the heavy fermion regime. Similarly, one can argue that condition (14) fits in with the heavy Higgs scenario ($\mathcal{M}_1 \ll \mathcal{M}_2$).

At *one-loop* the two sets of beta functions in $\overline{\text{2MS}}$ are given by

$$\begin{aligned}
&{}_i\beta^{(\text{1-loop})}_{g^2} = \varepsilon_i g^4, \quad {}_i\beta^{(\text{1-loop})}_{\lambda} = \alpha_i\lambda^2 + \eta_i\lambda g^2 + \zeta_i g^4, \quad {}_i\beta^{(\text{1-loop})}_{m^2} = m^2(\beta_i\lambda + \vartheta_i g^2), \\
&{}_i\beta^{(\text{1-loop})}_{\Lambda} = \gamma_i m^4, \quad {}_1\beta^{(\text{1-loop})}_{\varphi} = \iota_i g^2\varphi,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
&\alpha_1 = \frac{3\hbar}{(4\pi)^2}, \beta_1 = \frac{\hbar}{(4\pi)^2}, \gamma_1 = \frac{\hbar}{2(4\pi)^2}, \\
&\eta_2 = \frac{8\hbar N}{(4\pi)^2}, \zeta_2 = -\frac{48\hbar N}{(4\pi)^2}, \vartheta_2 = \frac{4\hbar N}{(4\pi)^2}, \iota_2 = \frac{2\hbar N}{(4\pi)^2}; \\
&\eta_1 = \zeta_1 = \vartheta_1 = \iota_1 = 0; \quad \alpha_2 = \beta_2 = \gamma_2 = 0.
\end{aligned} \tag{16}$$

The values of these constants are fixed by the single scale limit conditions iii) together with eqn. (10). However, these conditions are *not sufficient* to fix the values of ε_1 and ε_2. From (10) and (11) we know that $\varepsilon_1 + \varepsilon_2 = (4N+6)\hbar(4\pi)^{-2}$. The large-$N$ limit condition forces us to ascribe the N-dependent portion (ie. $4N\hbar(4\pi)^{-2}$) entirely to the second scale, so it only remains to "share" out the N-independent piece between the

[4]In ref. [5] we considered the more general possibility of a general linear combination of the beta functions $\tilde{\beta}_. = p_1\beta_. + (1-p)_2\beta_.$ taken to be independent of κ_1/κ_2. However, this RG had pathological properties except in the cases $p = 1$ or $p = 0$.

two scales. If the Higgs is heavy we would expect it to decouple from the low energy (fermionic) theory. Thus, the second set of beta functions should have contributions purely from fermionic loops. This fixes $\varepsilon_2 = 4N\hbar(4\pi)^{-2}$, and so we have $\varepsilon_1 = 6\hbar(4\pi)^{-2}$. Similarly, one can argue that for the heavy fermion case $\varepsilon_1 = 0$, and so $\varepsilon_2 = (4N+6)\hbar(4\pi)^{-2}$. To summarize the appropriate values of the ε_i are

$$\varepsilon_1 = \frac{6\hbar}{(4\pi)^2}, \quad \varepsilon_2 = \frac{4N\hbar}{(4\pi)^2} \quad (\mathcal{M}_1 \ll \mathcal{M}_2)$$

$$\varepsilon_1 = 0, \qquad \varepsilon_2 = \frac{(4N+6)\hbar}{(4\pi)^2} \quad (\mathcal{M}_2 \ll \mathcal{M}_2). \tag{17}$$

If one were to attempt to use these one-loop $\overline{2\mathrm{MS}}$ beta functions to produce two-scale running couplings via eqn. (9) one would run into an inconsistency, namely if one simply uses the one-loop beta functions given by (15) the integrability condition eqn. (12) is not satisfied. The point is that if we wish to consistently use the RG to sum up logarithms then we should first sum up the $\log(\kappa_1/\kappa_2)$-terms in the beta functions. To achieve this we select a subsidiary condition, in this case (13) or (14). If we select (13) then the first set of beta functions has no logarithms and so we only have to resum the logarithms in the second set of beta functions. To do this we use the integrability condition (12) which generates RG-type equations for the ${}_2\beta_{\cdot}$'s. As boundary conditions we just take the one-loop beta functions at $\kappa_1 = \kappa_2$, ie. ${}_2\beta_{\cdot}(\kappa_1 = \kappa_2) = {}_2\beta_{\cdot}^{(\text{1-loop})}$.

4 The LO resummed $\overline{2\mathrm{MS}}$ RG functions

4.1 First subsidiary condition, heavy fermion case

We now perform the LO resummation of the beta functions. First, let us deal with the case of the subsidiary condition (13). As $\mathcal{D}_1$ is now fixed eqn. (12) yields RG-type equations for the ${}_2\beta_{\cdot}$ which we have to solve. Setting

$$t = \frac{\hbar}{(4\pi)^2}\log\frac{\kappa_1}{\kappa_2} \tag{18}$$

the equation for ${}_2\beta_{g^2}$ becomes to leading order

$$\frac{\partial}{\partial t}\,{}_2\beta_{g^2}^{(\mathrm{LL})} + {}_1\beta_{g^2}^{(\mathrm{LL})}\frac{\partial}{\partial g^2}\,{}_2\beta_{g^2}^{(\mathrm{LL})} - {}_2\beta_{g^2}^{(\mathrm{LL})}\frac{\partial}{\partial g^2}\,{}_1\beta_{g^2}^{(\mathrm{LL})} = 0. \tag{19}$$

The solution does not explicitly depend on t

$${}_2\beta_{g^2}^{(\mathrm{LL})}(t) = \varepsilon_2 g^4. \tag{20}$$

We turn to the equation for ${}_2\beta_\lambda$

$$\frac{\partial}{\partial t}\,{}_2\beta_{\lambda}^{(\mathrm{LL})} + {}_1\beta_{g^2}^{(\mathrm{LL})}\frac{\partial}{\partial g^2}\,{}_2\beta_{\lambda}^{(\mathrm{LL})} - {}_2\beta_{g^2}^{(\mathrm{LL})}\frac{\partial}{\partial g^2}\,{}_1\beta_{\lambda}^{(\mathrm{LL})}$$
$$+{}_1\beta_{\lambda}^{(\mathrm{LL})}\frac{\partial}{\partial \lambda}\,{}_2\beta_{\lambda}^{(\mathrm{LL})} - {}_2\beta_{\lambda}^{(\mathrm{LL})}\frac{\partial}{\partial \lambda}\,{}_1\beta_{\lambda}^{(\mathrm{LL})} = 0. \tag{21}$$

This is easily solved via the method of characteristics

$$
{}_2\beta_\lambda^{(\mathrm{LL})}(t) = \lambda^2 \left(\eta_2 \, \frac{G_a(t)}{L_a(t)} + \zeta_2 \, \left(\frac{G_a(t)}{L_a(t)} \right)^2 \right) . \tag{22}
$$

Here, we have introduced the short-hand notations

$$
G_a(t) = \frac{g^2}{1+\varepsilon_1 g^2 t}, \quad L_a(t) = \frac{\lambda}{1+\alpha_1 \lambda t}. \tag{23}
$$

The remaining three β-functions can be determined in a similar fashion and we just quote the results

$$
\begin{aligned}
{}_2\beta_{m^2}^{(\mathrm{LL})}(t) &= m^2 L_a(t) \left(\beta_2 + \vartheta_2 \, \frac{G_a(t)}{L_a(t)} + \beta_1 \, \frac{t}{\lambda} \, {}_2\beta_\lambda^{(\mathrm{LL})}(t) \right) \\
{}_2\beta_\Lambda^{(\mathrm{LL})}(t) &= m^4 \frac{2\gamma_1}{2\beta_1 - \alpha_1} \left\{ \frac{1}{2} \left(\eta_2 \, \frac{G_a(t)}{L_a(t)} + \zeta_2 \, \left(\frac{G_a(t)}{L_a(t)} \right)^2 \right) \left(\frac{L_a(t)}{\lambda} \right)^{\frac{\beta_1}{\alpha_1}} \right. \\
&\quad \left. - \left(\beta_2 + \vartheta_2 \, \frac{G_a(t)}{L_a(t)} \right) \left(\frac{L_a(t)}{\lambda} \right)^{\frac{\beta_1}{\alpha_1}} - \frac{1}{2\lambda^2} \, {}_2\beta_\lambda^{(\mathrm{LL})}(t) + \frac{1}{m^2 \lambda} \, {}_2\beta_{m^2}^{(\mathrm{LL})}(t) \right\} \\
{}_2\beta_\varphi^{(\mathrm{LL})}(t) &= -\varphi \, \iota_2 G_a(t).
\end{aligned} \tag{24}
$$

The beta functions given here are suited to the heavy fermion regime.

4.2 Second subsidiary condition, heavy Higgs case

The LO beta functions associated with the second subsidiary condition (14) are obtained in the same way. We find the results

$$
\begin{aligned}
{}_1\beta_{g^2}^{(\mathrm{LL})}(t) &= \varepsilon_1 g^4 \\
{}_1\beta_\lambda^{(\mathrm{LL})}(t) &= \left(\alpha_1 L_b(t)^2 - \frac{\varepsilon_1}{\varepsilon_2} \eta_2 G_b(t) L_b(t) - \frac{\varepsilon_1}{\varepsilon_2} \zeta_2 G_b(t)^2 \right) \left(\frac{G_b(t)}{g^2} \right)^{-\frac{\eta_2}{\varepsilon_2}} \\
&\quad + \left(\frac{\eta_2}{\varepsilon_2} \frac{\lambda}{g^2} + \frac{\zeta_2}{\varepsilon_2} \right) {}_1\beta_{g^2}^{(\mathrm{LL})}(t).
\end{aligned} \tag{25}
$$

Here, we have introduced the short-hand notations

$$
\begin{aligned}
G_b(t) &= \frac{g^2}{1-\varepsilon_2 g^2 t}, \\
L_b(t) &= \left(\frac{G_b(t)}{g^2} \right)^{\frac{\eta_2}{\varepsilon_2}} \left(\lambda - \frac{\zeta_2}{\varepsilon_2 - \eta_2} g^2 \right) + \frac{\zeta_2}{\varepsilon_2 - \eta_2} G_b(t).
\end{aligned} \tag{26}
$$

As for the remaining β-functions we find

$$
\begin{aligned}
{}_1\beta_{m^2}^{(\mathrm{LL})}(t) &= m^2 \left(\beta_1 L_b(t) - \frac{\vartheta_2}{\varepsilon_2} G_b(t) \right) + m^2 \frac{\vartheta_2}{\varepsilon_2} \frac{1}{g^2} {}_1\beta_{g^2}^{(\mathrm{LL})}(t), \\
{}_1\beta_\Lambda^{(\mathrm{LL})}(t) &= m^4 \gamma_1 \left(\frac{G_b(t)}{g^2} \right)^{2\frac{\vartheta_2}{\varepsilon_2}}, \\
{}_1\beta_\varphi^{(\mathrm{LL})}(t) &= -\varphi \, \frac{\varepsilon_1}{\varepsilon_2} \iota_2 \left(g^2 - G_b(t) \right).
\end{aligned} \tag{27}
$$

These RG functions are suitable for the heavy Higgs case, ie. t large and positive. It is clear that the LO beta functions given here possess a Landau pole type singularity for positive t. Thus it seems we cannot take t to be too large, and we have apparently failed to access the heavy Higgs regime. However, when computing the improved potential we always use the *running* couplings, and if λ and g^2 run to zero sufficiently quickly in the limit $\kappa_1 \to \infty$ then the Landau pole may never be reached. It is amusing to note that one can take the limit $t \to -\infty$ without difficulty, whereas decoupling arguments indicate that this is inappropriate for the second subsidiary condition.

5 The LO $\overline{\text{2MS}}$ running two-scale parameters

We now work out the two-scale running parameters. The running parameters in $\overline{\text{2MS}}$ are functions of the variables

$$s_i = \frac{\hbar}{(4\pi)^2} \log \frac{\kappa_i(s_i)}{\kappa_i}, \quad t = \frac{\hbar}{(4\pi)^2} \log \frac{\kappa_1}{\kappa_2}, \tag{28}$$

where κ_i are the reference scales. They may be expanded in series in $\hbar$ the LO terms of which we determine now from eqn. (9).

The equations for the leading order running two-scale coupling are

$$\frac{dg^{2(\text{LL})}}{ds_i} = \varepsilon_i \left(g^{2(\text{LL})}\right)^2. \tag{29}$$

They are easily integrated and the result applies to both subsidiary conditions

$$g^{2(\text{LL})}(s_i) = \frac{g^2}{1 - g^2(\varepsilon_1 s_1 + \varepsilon_2 s_2)} \tag{30}$$

with the boundary condition $g^{2(\text{LL})}(s_i = 0) = g^2$.

5.1 First subsidiary condition

We turn to the computations specifically associated with our first subsidiary condition (13). Starting with the running λ we first solve

$$\frac{d{\lambda_a}^{(\text{LL})}}{ds_1} = \alpha_1 \lambda_a^{(\text{LL})2} \tag{31}$$

with the result

$${\lambda_a}^{(\text{LL})}(s_i) = \frac{1}{C_a(s_2) - \alpha_1 s_1}. \tag{32}$$

Here, $C_a(s_2)$ is the constant of integration which is determined from the second λ-equation

$$\frac{d{\lambda_a}^{(\text{LL})}}{ds_2} = \left(\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)} \eta_2 + \left(\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)} \right)^2 \zeta_2 \right) \lambda_a^{(\text{LL})2}. \tag{33}$$

For convenience we have introduced the functions $\tilde{G}_a(s_2), \tilde{L}_a(s_2)$ obtained from eqns. (23) after insertion of the respective LO results (30), (32)

$$\begin{aligned} \tilde{G}_a(s_2) &= \frac{g^2}{1-g^2\varepsilon\, s_2 + g^2\varepsilon_1 t}, \quad \text{where} \quad \varepsilon = \varepsilon_1 + \varepsilon_2, \\ \tilde{L}_a(s_2) &= \frac{1}{C(s_2) - \alpha_1 s_2 + \alpha_1 t}. \end{aligned} \tag{34}$$

We may now express the running λ completely in terms of $\tilde{L}_a(s_2)$

$$\lambda_a{}^{(\mathrm{LL})}(s_i) = \frac{\tilde{L}_a(s_2)}{1-\tilde{L}_a(s_2)\,\alpha_1(s_1 - s_2 + t)}. \tag{35}$$

Hence, the remaining task is to determine $\tilde{L}_a(s_2)$. Insertion of (35) into (33) yields an ODE in the variable s_2 for $\tilde{L}_a(s_2)$

$$\frac{d\tilde{L}_a}{ds_2} = \alpha_1 \tilde{L}_a^2(s_2) + \eta_2 \tilde{G}_a(s_2)\tilde{L}_a(s_2) + \zeta_2 \tilde{G}_a^2(s_2) \tag{36}$$

which is solved by

$$\tilde{L}_a(s_2) = \tilde{G}_a(s_2)\, \frac{\rho_a - \sigma_a K_a \left(\frac{\tilde{G}_a(s_2)}{\tilde{G}_a}\right)^{\frac{\alpha_1(\rho_a-\sigma_a)}{\varepsilon}}}{1-K_a\left(\frac{\tilde{G}_a(s_2)}{\tilde{G}_a}\right)^{\frac{\alpha_1(\rho_a-\sigma_a)}{\varepsilon}}}. \tag{37}$$

Here, we have introduced

$$K_a = \frac{y-\rho_a}{y-\sigma_a}, \quad y = \frac{\tilde{L}_a}{\tilde{G}_a}, \tag{38}$$

where $\tilde{L}_a = \tilde{L}_a(s_2 = 0)$ and $\tilde{G}_a = \tilde{G}_a(s_2 = 0)$. ρ_a and σ_a are the two roots of the quadratic equation $\alpha_1 z^2 + (\eta_2 - \varepsilon)z + \zeta_2 = 0$, hence

$$\rho_a/\sigma_a = -\frac{\eta_2 - \varepsilon}{2\alpha_1} +/- \frac{1}{2\alpha_1}\sqrt{\left(\frac{\eta_2-\varepsilon}{2}\right)^2 - 4\zeta_2\alpha_1} \tag{39}$$

The boundary condition is chosen such that $\lambda_a{}^{(\mathrm{LL})}(s_i = 0) = \lambda$ requiring $\tilde{L}_a(s_2 = 0) = \frac{\lambda}{1+\lambda\alpha_1 t}$.

Next we determine the running mass from

$$\frac{dm_a^{2\,(\mathrm{LL})}}{ds_1} = \beta_1 \lambda_a^{(\mathrm{LL})} m_a^{2\,(\mathrm{LL})}. \tag{40}$$

This is easily solved

$$m_a^{2\,(\mathrm{LL})}(s_i) = m^2 D_a(s_2) \left(\frac{\lambda_a{}^{(\mathrm{LL})}(s_i)}{\lambda}\right)^{\frac{\beta_1}{\alpha_1}}. \tag{41}$$

The constant of integration $D_a(s_2)$ has to be obtained from the second m^2-equation

$$\frac{dm_a^{2\,(\mathrm{LL})}}{ds_2} = m_a^{2\,(\mathrm{LL})} \tilde{L}_a(s_2)\left(\beta_2 + \frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)}\,\vartheta_2 + \frac{s_1 - s_2 + t}{\lambda_a{}^{(\mathrm{LL})}}\; {}_2\beta_\lambda^{(\mathrm{LL})}\;\beta_1\right). \tag{42}$$

Inserting (41) into (42) yields the sought-after ODE for $D_a(s_2)$

$$\frac{1}{D_a}\frac{dD_a}{ds_2} = \tilde{L}_a(s_2)\left(\beta_2 + \left(\vartheta_2 - \frac{\beta_1}{\alpha_1}\eta_2\right)\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)} - \frac{\beta_1}{\alpha_1}\zeta_2\left(\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)}\right)^2\right). \tag{43}$$

The integrations are a bit cumbersome but straightforward and we obtain

$$\begin{aligned} D_a(s_2) \; = \; & \left(\frac{\tilde{G}_a(s_2)}{\tilde{G}_a}\right)^{\frac{\rho_a\beta_2}{\varepsilon}+\frac{1}{\varepsilon}\left(\vartheta_2-\frac{\beta_1}{\alpha_1}\eta_2\right)-\frac{1}{\varepsilon\rho_a}\frac{\beta_1}{\alpha_1}\zeta_2} \\ & \left(\frac{\rho_a-\sigma_a K_a\left(\frac{\tilde{G}_a(s_2)}{\tilde{G}_a}\right)^{\frac{\alpha_1(\rho_a-\sigma_a)}{\varepsilon}}}{\rho_a-\sigma_a K_a}\right)^{-\frac{\beta_1}{\alpha_1}}\left(\frac{1-K_a\left(\frac{\tilde{G}_a(s_2)}{\tilde{G}_a}\right)^{\frac{\alpha_1(\rho_a-\sigma_a)}{\varepsilon}}}{1-K_a}\right)^{-\frac{\beta_2}{\alpha_1}} . \end{aligned} \tag{44}$$

The boundary condition is $m_a^{2\,(\mathrm{LL})}(s_i=0)=m^2$.

In order to obtain the running cosmological constant we have to solve

$$\frac{d\Lambda_a{}^{(\mathrm{LL})}}{ds_1} = \gamma_1\left(m_a^{2\,(\mathrm{LL})}\right)^2 . \tag{45}$$

This yields the result

$$\Lambda_a{}^{(\mathrm{LL})}(s_i) = \frac{\gamma_1}{2\beta_1-\alpha_1}\frac{\left(m_a^{2\,(\mathrm{LL})}(s_i)\right)^2}{\lambda_a{}^{(\mathrm{LL})}(s_i)} + E_a(s_2). \tag{46}$$

To calculate the constant of integration $E_a(s_2)$ we turn to the second Λ-equation

$$\begin{aligned} \frac{d\Lambda_a{}^{(\mathrm{LL})}}{ds_2} = \left(m_a^{2\,(\mathrm{LL})}\right)^2 \frac{2\gamma_1}{2\beta_1-\alpha_1} & \left\{\frac{1}{2}\left(\eta_2\,\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)} + \zeta_2\left(\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)}\right)^2\right)\left(\frac{\tilde{L}_a(s_2)}{\lambda^{(\mathrm{LL})}}\right)^{\frac{\beta_1}{\alpha_1}}\right. \\ & \left. - \left(\beta_2 + \vartheta_2\,\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)}\right)\left(\frac{\tilde{L}_a(s_2)}{\lambda_a{}^{(\mathrm{LL})}}\right)^{\frac{\beta_1}{\alpha_1}} - \frac{1}{2\lambda_a{}^{(\mathrm{LL})2}}\,2\beta_\lambda^{(\mathrm{LL})} + \frac{1}{\lambda_a{}^{(\mathrm{LL})}}\,2\beta_{m^2}^{(\mathrm{LL})}\right\}. \end{aligned} \tag{47}$$

The ODE for $E_a(s_2)$ is obtained by inserting (46) into (47)

$$\frac{dE_a}{ds_2} = \frac{\gamma_1}{2\beta_1-\alpha_1}\left(-2\beta_2 + (\eta_2-2\vartheta_2)\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)} + \zeta_2\left(\frac{\tilde{G}_a(s_2)}{\tilde{L}_a(s_2)}\right)^2\right) D_a(s_2)^2\,\tilde{L}_a(s_2)^{\frac{2\beta_1}{\alpha_1}} . \tag{48}$$

The integrations above may not be performed in a closed form.

The running field does depend only on s_2

$$\frac{d\varphi_a{}^{(\mathrm{LL})}}{ds_2} = -\iota_2\,\varphi_a{}^{(\mathrm{LL})}\tilde{G}_a(s_2) \tag{49}$$

and we find

$$\varphi_a{}^{(\mathrm{LL})}(s_i) = \varphi\left(\frac{\tilde{G}_a(s_2)}{\tilde{G}_a}\right)^{\frac{\iota_2}{\varepsilon}} \tag{50}$$

obeying the boundary condition $\varphi_a{}^{(\mathrm{LL})}(s_i=0)=\varphi$.

5.2 Second subsidiary condition

We next quote the results for the couplings corresponding to our second subsidiary condition (13).

$$\lambda_b{}^{(\mathrm{LL})}(s_i) = \frac{\zeta_2}{\varepsilon_2 - \eta_2} g^{2(\mathrm{LL})}(s_i) + \left(g^{2(\mathrm{LL})}(s_i)\right)^{\frac{\eta_2}{\varepsilon_2}} C_b(s_1). \tag{51}$$

To proceed further it is convenient to introduce the function $\tilde{G}_b(s_1)$ obtained from eqn. (26) after insertion of the LO result (30)

$$\tilde{G}_b(s_1) = \frac{g^2}{1 - g^2\varepsilon\, s_1 + g^2\varepsilon_2 t}, \quad \text{where} \quad \varepsilon = \varepsilon_1 + \varepsilon_2. \tag{52}$$

Again we set $\tilde{G}_b = \tilde{G}_b(s_1 = 0)$. Then the function $C_b(s_1)$ is given by

$$C_b(s_1) = \left(\tilde{G}_b(s_1)\right)^{1-\frac{\eta_2}{\varepsilon_2}} \frac{\rho_b - \sigma_b K_b \left(\frac{\tilde{G}_b(s_1)}{\tilde{G}_b}\right)^{\frac{\alpha_1(\rho_b-\sigma_b)}{\varepsilon}}}{1 - K_b \left(\frac{\tilde{G}_b(s_1)}{\tilde{G}_b}\right)^{\frac{\alpha_1(\rho_b-\sigma_b)}{\varepsilon}}}. \tag{53}$$

The constant of integration is

$$K_b = \frac{y - \rho_b}{y - \sigma_b}, \quad y = \left(g^2\right)^{-\frac{\eta_2}{\varepsilon_2}} \left(\tilde{G}_b\right)^{\frac{\eta_2}{\varepsilon_2}-1} \left(\lambda - \frac{\zeta_2}{\varepsilon_2 - \eta_2} g^2\right) \tag{54}$$

and ρ_b and σ_b are the two roots of the quadratic equation $\alpha_1 z^2 + (2\alpha_1 \frac{\zeta_2}{\varepsilon_2 - \eta_2} + \eta_2 - \varepsilon) z + \alpha_1 \left(\frac{\zeta_2}{\varepsilon_2-\eta_2}\right)^2 - \varepsilon_1 \frac{\zeta_2}{\varepsilon_2-\eta_2} = 0$, hence

$$\rho_b/\sigma_b = -\frac{\zeta_2}{\varepsilon_2 - \eta_2} - \frac{\eta_2 - \varepsilon}{2\alpha_1} +/- \frac{1}{2\alpha_1}\sqrt{(\eta_2 - \varepsilon)^2 - 4\zeta_2\alpha_1}. \tag{55}$$

The boundary condition is chosen such that $\lambda_b{}^{(\mathrm{LL})}(s_i = 0) = \lambda$.

The running mass is found to be

$$m_b^{2(\mathrm{LL})}(s_i) = m^2 D_b(s_1) \left(\frac{g^{2(\mathrm{LL})}(s_i)}{g^2}\right)^{\frac{\vartheta_2}{\varepsilon_2}}, \tag{56}$$

where

$$D_b(s_1) = \left(\frac{\tilde{G}_b(s_1)}{\tilde{G}_b}\right)^{\frac{\rho_b\beta_1}{\varepsilon} + \frac{1}{\varepsilon}\left(-\vartheta_2\frac{\varepsilon_1}{\varepsilon_2} + \beta_1\frac{\zeta_2}{\varepsilon_2-\eta_2}\right)} \left(\frac{1 - K_b\left(\frac{\tilde{G}_b(s_1)}{\tilde{G}_b}\right)^{\frac{\alpha_1(\rho_b-\sigma_b)}{\varepsilon}}}{1 - K_b}\right)^{-\frac{\beta_1}{\alpha_1}}. \tag{57}$$

The boundary condition is $m_b^{2(\mathrm{LL})}(s_i = 0) = m^2$.

The running cosmological constant is only dependent on s_1 and fulfils the equation

$$\frac{d\Lambda_b{}^{(\mathrm{LL})}}{ds_1} = \gamma_1 m^4 \left(\frac{\tilde{G}_b(s_1)}{\tilde{G}_b}\right)^{\frac{2\rho_b\beta_1}{\varepsilon} + \frac{2}{\varepsilon}\left(-\vartheta_2\frac{\varepsilon_1}{\varepsilon_2} + \beta_1\frac{\zeta_2}{\varepsilon_2-\eta_2}\right)} \left(\frac{1 - K_b\left(\frac{\tilde{G}_b(s_1)}{\tilde{G}_b}\right)^{\frac{\alpha_1(\rho_b-\sigma_b)}{\varepsilon}}}{1 - K_b}\right)^{-\frac{2\beta_1}{\alpha_1}} \left(g^2\tilde{G}_b(s_1)\right)^{-\frac{2\vartheta_2}{\varepsilon_2}}. \tag{58}$$

The integration above yields a hypergeometric function.

The running field is found to be

$$
{\varphi_b}^{(\mathrm{LL})}(s_i) = \varphi \left(\frac{\tilde{G}_b(s_1)}{\tilde{G}_b} \right)^{-\frac{\varepsilon_1}{\varepsilon_2}\frac{\iota_2}{\varepsilon}} \left(g^{2^{(\mathrm{LL})}}(s_i) \right)^{-\frac{\iota_2}{\varepsilon_2}} \tag{59}
$$

and obeys the boundary condition ${\varphi_b}^{(\mathrm{LL})}(s_i = 0) = \varphi$.

6 LO RG Improved Potential

The LO RG improved potential for both subsidiary conditions is simply

$$
V^{(\mathrm{LL})} = \frac{\lambda^{(\mathrm{LL})}(s_i)}{24} \varphi^{(\mathrm{LL})}(s_i)^4 + \frac{m^{2^{(\mathrm{LL})}}(s_i)}{2} \varphi^{(\mathrm{LL})}(s_i)^2 + \Lambda^{(\mathrm{LL})}(s_i), \tag{60}
$$

where

$$
s_i = \frac{\hbar}{2(4\pi)^2} \log \frac{\mathcal{M}_i}{\kappa_i^2}. \tag{61}
$$

7 Conclusions

Using a two-scale RG we have studied the effective potential for the Higgs-Yukawa model. We have *two* improved potentials, based on the two subsidiary conditions (13) and (14). Using decoupling arguments it has been argued that these prescriptions are appropriate for the heavy fermion and heavy Higgs case, respectively. By construction both these potentials reduce to the standard CW improved potential in the single-scale limits. Accordingly, we expect that our first improved potential interpolates from the heavy fermion regime to the single scale-case ($\mathcal{M}_1 \approx \mathcal{M}_2$). Similarly our second improved potential should interpolate between the single-scale regime and the heavy Higgs case. It is not clear whether it possible to devise a scheme which correctly interpolates all the way from the heavy fermion to the heavy Higgs problem. It may be that a variant of the scheme presented in [3] will do this, since in these schemes one does not have to select a subsidiary condition. However, it is quite difficult to compute the improved potential in such schemes [4], and it is not known whether this variety of schemes correctly incorporates the decoupling of heavy particles. In general, it would be useful to have a scheme where one did not have to assume decoupling, That is, one could *derive* whether decoupling did or did not occur starting from a simple multi-scale scheme.

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